OSCILLATION OF A CONDUCTING GAS IN A PERIODIC ELECTROMAGNETIC FIELD

L. A. Zaklyaz'minskii and A. N. Cherepanov

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It is assumed that the source of the conducting gas is the plane $x = 0$, the gas moving along the x-axis. At distances $x = \pm l$ from the source lie conducting planes, in which the currents flow along the y-axis into an external load R and are driven by an external emf $E°$. These planes are absolutely permeable to the gas, so the magnetic field set up by the currents is directed along the z-axis, along which also lies a steady magnetic field H_0. The system is unbounded in the x- and y-directions, so we will consider an external circuit consisting of resistance R per unit height and an external emf $E°$ applied to a part of the plane of unit height and of width l in y. The following assumptions are made:

1) The gas conductivity is finite and is dependent on the temperature T:

$$\sigma / \sigma_0 = (T / T_0)^n \quad (n \geqslant 0). \tag{1}$$

2) The gas is ideal, and thermal conductivity and viscosity are neglected.

3) The magnetohydrodynamics approximation applies.

Here the following are the dimensionless equations for the motion of the gas, the distribution of the magnetic field in the medium, and the current in the external circuit:

$$\varkappa M_0^2 g\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}\right)=-\frac{\partial f}{\partial \xi}-\mathrm{P}h\frac{\partial h}{\partial \xi},\quad \frac{\partial g}{\partial \tau}+\frac{\partial}{\partial \xi}(gu)=0,$$

$$\frac{g}{\varkappa-1}\left(\frac{\partial \theta}{\partial \tau}+u\frac{\partial \theta}{\partial \xi}\right)+g\theta\frac{\partial u}{\partial \xi}=-\mathrm{P}\varepsilon\frac{\partial h}{\partial \xi}+\mathrm{P}uh\frac{\partial h}{\partial \xi},\quad f=g\theta,$$

$$\frac{\partial h}{\partial \xi}=R_m\sigma_1(uh-\varepsilon),\quad \frac{\partial \varepsilon}{\partial \xi}=-\frac{\partial h}{\partial \tau},\quad \frac{\partial}{\partial \tau}\int_0^1 h\,d\xi+ri=E_1(\tau), \tag{2}$$

in which the dimensionless quantities are defined as follows:

$$g=\frac{\rho}{\rho_0},\quad u=\frac{v}{v_0},\quad f=\frac{P}{P_0},\quad \theta=\frac{T}{T_0},\quad h=\frac{H}{H_0},$$

$$\sigma_1=\frac{\sigma}{\sigma_0},\quad \varepsilon=\frac{cE}{v_0H_0},\quad E_1=\frac{cE^\circ}{v_0H_0},\quad i=\frac{4\pi I}{cH_0},\quad r=\frac{c^2R}{4\pi u_0},$$

$$\xi=\frac{x}{l},\quad \tau=\frac{v_0t}{l},\quad \varkappa=\frac{c_p}{c_v},\quad M_0=\frac{v_0}{\sqrt{\varkappa P_0/\rho_0}},$$

$$\mathrm{P}=\frac{H_0^2}{4\pi P_0},\quad R_m=\frac{4\pi\sigma_0 v_0 l}{c^2}.$$

The scales for the physical variables are ρ_0, v_0, P_0, and T_0 (the maximum values of these quantities at the exit from the source), H_0 (the constant component of the external magnetic field), σ_0 (the conductivity at T_0), and l.

The velocity has only an x-component, the electric field E only a y-component, and the magnetic field H only a z-component.

The boundary conditions for the gasdynamic quantities are set by the source output, while those for the electromagnetic quantities are set by the conditions for reflection of perturbations at $\xi = 0.2$:

$$\varepsilon(0,\tau)=0,\quad h(1,\tau)=1+i(\tau). \tag{3}$$

We seek a periodic solution to (2) subject to the condition that the external emf $E_1(\tau)$ is a periodic function of time. We consider the case in which the gasdynamic quantities at the exit from the source are constant:

$$f=g=u=1 \quad \text{for} \quad \xi=0. \tag{4}$$

We solve (2) with (3) and (4) as power series in ξ:

$$f(\xi,\tau)=1+\sum_{k=1}^{m}f_k(\tau)\xi^k,\quad g(\xi,\tau)=1+\sum_{k=1}^{m}g_k(\tau)\xi^k,$$

$$\theta(\xi,\tau)=1+\sum_{k=1}^{m}\theta_k(\tau)\xi^k,\quad u(\xi,\tau)=1+\sum_{k=1}^{m}u_k(\tau)\xi^k,$$

$$h(\xi,\tau)=\sum_{k=0}^{m}h_k(\tau)\xi^k,\quad \varepsilon(\xi,\tau)=-\sum_{k=0}^{m}\frac{\partial h_k}{\partial \tau}\frac{\xi^{k+1}}{k+1}. \tag{5}$$

Here m is an integer defining the order of approximation. The expression for $\varepsilon(\xi,\tau)$ is derived from the sixth equation of (2) with allowance for (3).

We substitute into (2) with $n = 1$ the expressions of (5) for f, g, θ, u, h, and ε, and equate terms with equal powers of ξ.

If we restrict the series of (5) to $m = 2$, we have the following relations for the time coefficients:

$$h_1=R_mh_0,\quad h_2=\frac{R_m}{2}\left[h_0'+R_mh_0+\frac{1+M_0^2(1-\varkappa)}{1-M_0^2}Sh_0^3\right],$$

$$S=\mathrm{P}R_m,\qquad f_1=\frac{M_0^2(1-\varkappa)-1}{1-M_0^2}Sh_0^2,$$

$$f_2=\frac{M_0^2(\varkappa-1)+\varkappa}{(1-M_0^2)^2}S^2h_0^4+\frac{S}{1-M_0^2}h_0h_0'-u_2+\theta_2,$$

$$\theta_1=M_0^2\frac{1-\varkappa}{1-M_0^2}Sh_0^2,$$

$$\theta_2=-\frac{1+\varkappa M_0^2}{1-M_0^2}Sh_0h_0'-\frac{SR_m}{2}h_0^2-\frac{1+M_0^2(\varkappa-1)}{(1-M_0^2)^2}S^2h_0^4-$$

$$-\mathrm{P}h_0h_2+(1-\varkappa M_0^2)u_2,\quad u_1=\frac{S}{1-M_0^2}h_0^2,$$

$$u_2=\frac{2\varkappa+1+M_0^2(\varkappa-1)}{2\varkappa(1-M_0^2)}Sh_0h_0'+\frac{1}{1-M_0^2}SR_mh_0^2+$$

$$+\frac{3+M_0^2[\varkappa-2+(1-\varkappa)(1-M_0^2)]}{2(1-M_0^2)^2}S^2h_0^4, \tag{6}$$

with the prime denoting differentiation with respect to τ.

We see from (6) that all the coefficients in the series of (5) are expressed in terms of $h_0(\tau)$ (the magnetic field in the plane $\xi = 0$) and derivatives of this. The solution is evidently meaningful for $M_0 < 1$ or $M_0 > 1$.

The equations for $h_0(\tau)$ are derived from the last equation of (2) together with (3), the result being

$$h_0''+ph_0'+qh_0=\frac{6}{R_m}(r+E_1)-S\varphi(h_0,h_0'), \tag{7}$$

$$\varphi(h_0,h_0')=3\frac{1+M_0^2(1-\varkappa)}{1-M_0^2}h_0^2(h_0'+rh_0),$$

$$p=\frac{6}{R_m}\left[1+\frac{R_m}{2}(1+r)+\frac{R_m^2}{6}\right],$$

$$q=\frac{6r}{R_m}\left(1+R_m+\frac{R_m^2}{2}\right). \tag{8}$$

We write E_1 as

$$E_1=E_0\sin\omega\tau,\qquad \omega=\Omega l/v_0,$$

in which E_0 is the dimensionless amplitude and ω is the dimensionless frequency.

We seek a periodic solution to (7) by successive approximation, on the assumption that $S \ll 1$ in the nonlinear expression $\varphi(h_0, h_0')$. The iteration formula can be written as

$$h_{0\,k+1} = \frac{6}{R_m}\left[\frac{r}{q} + \right.$$

$$\left. + \frac{E_0}{\sqrt{(q-\omega^2)^2 + p^2\omega^2}} \sin\left(\omega\tau + \operatorname{arc\,tg} \frac{q-\omega^2}{p\omega}\right)\right] +$$

$$+ \frac{S}{\lambda}\left[e^{\alpha_1\tau}\int \varphi(h_{0k}, h_{0k}')\, e^{-\alpha_1\tau}\, d\tau - e^{\alpha_2\tau}\int \varphi(h_{0k}, h_{0k}')\, e^{-\alpha_2\tau} d\tau\right],$$

$$\alpha_1 = {}^1/_2(\lambda - p), \quad \alpha_2 = -{}^1/_2(\lambda + p), \quad \lambda = \sqrt{p^2 - 4q}, \qquad (9)$$

in which k is the order of approximation.

The assumption $R_m \ll 1$ is involved in approximating the solution by three terms. More terms must be taken in (5) if the R_m increase.

The numerical analysis has been performed for the case $M_0 \ll 1$, with m = 3 in (5) (four terms). The time coefficients of these four terms are

$$h_3 = {}^1/_3 R_m ({}^1/_2 R_m^2 h_0 + R_m h_0' +$$

$$+ (\varkappa + 2)\varkappa^{-1} S h_0^2 h_0' + {}^5/_2 S R_m h_0^3 + {}^3/_2 S^2 h_0^5),$$

$$f_3 = -{}^1/_3 S \{{}^5/_2 R_m h_0 h_0' + 2R_m^2 h_0^2 +$$

$$+ S[(\varkappa + 2)\varkappa^{-1} h_0' + 4R_m h_0 + {}^3/_2 S h_0^3]\, h_0\},$$

$$\theta_3 = {}^1/_3 S (1 - \varkappa)\varkappa^{-1}[{}^1/_2 R_m h_0 h_0' - h_0'^2 + S h_0^3 h_0'],$$

$$u_3 = {}^1/_3 S\left(R_m h_0^2 + \frac{2\varkappa + 1}{2\varkappa} R_m h_0 h_0' + \frac{2\varkappa + 1}{6\varkappa} h_0'^2 + \right.$$

$$+ \frac{2\varkappa - 1}{2\varkappa} S h_0^3 h_0' + 6 S R_m h_0^4 +$$

$$\left. + {}^{15}/_2 S^2 h_0^6 + {}^1/_2 \varkappa h_0 h_0''\right).$$

We obtain the coefficients of the second and third terms from (6) if we neglect the terms containing M_0. The function $h_0(\tau)$ is defined by

$$h_0'' + p h_0' + q h_0 = \frac{12}{R_m(2 + R_m)}(r + E_1) - S\varphi(h_0, h_0', h_0''), \qquad (10)$$

in which

$$\varphi(h_0, h_0', h_0'') = \frac{6h_0}{2 + R_m} \times$$

$$\times \left[\frac{\varkappa + 2}{12\varkappa} h_0 h_0'' + \left(1 + {}^5/_4 R_m + \frac{\varkappa + 2}{3\varkappa} r R_m\right) \times \right.$$

$$\times h_0 h_0' + \frac{\varkappa + 2}{12\varkappa} h_0'^2 + (1 + {}^5/_3 R_m)\, r h_0^2 + S({}^5/_4 h_0' + r h_0)\, h_0^3,$$

$$p = \frac{6}{R_m}\,\frac{2 + R_m(1 + r) + {}^1/_3 R_m^2 (1 + 2r + {}^1/_4 R_m)}{2 + R_m}, \qquad (11)$$

$$q = \frac{6r}{R_m(2 + R_m)}(2 + 2R_m + R_m^2 + {}^1/_3 R_m^3). \qquad (12)$$

In the first and second approximations we have

$$h_{01}(\tau) = \frac{12}{R_m(2 + R_m)} \times$$

$$\times \left[\frac{r}{q} + \frac{E_0}{\sqrt{(q - \omega^2)^2 + p^2\omega^2}} \sin\left(\omega\tau + \operatorname{arctg}\frac{q - \omega^2}{p\omega}\right)\right], \qquad (13)$$

$$h_{02}(\tau) = \frac{12r}{R_m(2 + R_m)\, q} + S B_0 + B_1 \sin(\omega\tau + \delta_1) +$$

$$+ B_2 \sin(2\omega\tau + \delta_2) + B_3 \sin(3\omega\tau + \delta_3) \qquad (14)$$

Here

$$B_0 = {}^1/_2 (a_5 - a_4 + 2a_9)/q,$$

$$B_1 = \{[A_1 + S(A_2 + {}^3/_4 A_8)]^2 + S^2 (A_3 + {}^1/_4 A_9)^2\}^{1/2},$$

$$B_3 = S(A_4^2 + S_5^2)^{1/2}, \quad B_4 = S[(A_6 + {}^1/_4 A_8)^2 + (A_7 - {}^1/_4 A_9)^2]^{1/2},$$

$$\delta_1 = \operatorname{arc\,tg}\frac{q - \omega^2}{p\omega} + \operatorname{arc\,tg}\frac{S(A_3 + {}^1/_4 A_9)}{A_1 + S(A_2 + {}^3/_4 A_8)},$$

$$\delta_2 = \operatorname{arc\,tg}\frac{A_5}{A_4},$$

$$\delta_3 = \operatorname{arc\,tg}\frac{4A_7 - A_9}{4A_6 + A_8},$$

$$A_2 = \Omega_1^{-1}\{(q - \omega^2)^2 a_1 - \omega a_2 - 6\Omega_2^{-1}\omega^2 q\,[p^2 - q +$$

$$+ \omega^2(10q - 9\omega^2)]\, a_6 - {}^1/_2 \omega a_7 + {}^1/_2 (q - \omega^2)\, a_8\},$$

$$A_3 = \Omega_1^{-1}[\omega a_1 + (q - \omega^2)(a_2 + {}^1/_2 a_7) +$$

$$+ {}^1/_2 \omega a_8 + 6\Omega_2^{-1}(10\omega^2 + p^2 - 2q)\, p\omega^3 a_6],$$

$$A_4 = {}^1/_2 \Omega_3^{-1}[2(q - 4\omega^2)\, a_3 + p\omega(a_4 + a_5)],$$

$$A_5 = {}^1/_2 \Omega_3^{-1}[4\omega a_3 + (q - 4\omega^2)(a_4 + a_5)],$$

$$A_6 = {}^1/_2 \Omega_2^{-1}[3\omega a_7 + (q - 9\omega^2)\, a_8],$$

$$A_7 = {}^1/_2 \Omega_2^{-1}[3\omega a_8 - (q - 9\omega^2)\, a_7],$$

$$A_8 = \Omega_2^{-1}(q - 9\omega^2)\, a_6, \quad A_9 = 3\Omega_2^{-1} p\omega a_6,$$

$$\Omega_1 = (q - \omega^2)^2 + p^2\omega^2, \quad \Omega_2 = (q - 9\omega^2)^2 + 9p^2\omega^2,$$

$$\Omega_3 = (q - 4\omega^2)^2 + 4p^2\omega^2,$$

$$a_1 = \left[(3 + 5R_m)\, r - \frac{\varkappa + 2}{12\varkappa}\omega^2\right] a^2 A_1,$$

$$a_2 = \omega a^2 A_1\left(1 + {}^5/_4 R_m + \frac{\varkappa + 2}{3\varkappa} r R_m\right),$$

$$a_3 = \frac{a_2 A_1}{a}, \quad a_4 = a A_1^2\left[\frac{\varkappa + 2}{6\varkappa}\omega^2 - (3 + 5R_m)\, r\right],$$

$$a_6 = \left(\frac{\varkappa + 2}{12\varkappa}\omega^2 + \frac{3 + 5R_m}{3} r\right) A_1^3, \quad a_5 = \frac{\varkappa + 2}{12\varkappa}\omega^2 a A_1,$$

$$a_7 = \frac{\omega}{2} A_1^3\left(1 + \frac{5}{4} R_m + \frac{\varkappa + 2}{3\varkappa} r R_m\right),$$

$$a_8 = \frac{\varkappa + 2}{24\varkappa}\omega^2 A_1 \qquad a_9 = \frac{3 + 5R_m}{3} r a^3,$$

$$a = \frac{12r}{R_m(2 + R_m)\, q}, \quad A_1 = \frac{12E_0}{R_m(2 + R_m)\,\Omega_1^{1/2}}.$$

The quantities p and q are defined by (11) and (12), respectively. Consider the resonance condition, initially via the first-approximation solution of (13), which is the linear solution for the time variation in the magnetic field at $\xi = 0$.

It follows from (10) and (13) that the external driving force has the constant component $12r/R_m(2 + R_m)$ and the sinusoidal component $12E_0/R_m(2 + R_m)$, and it produces a constant deflection of $12r/R_m(2 + R_m)\, q$ together with a sinusoidal oscillation of the same frequency but a different amplitude and phase.

We divide the amplitude in the second term in (13) by the value for $\omega = 0$ to get the gain K, which characterizes the dynamic susceptibility of the system with respect to the external force:

$$K(\mu) = [(1 - \mu^2)^2 + \nu^2\mu^2]^{-1/2}$$

$$(\mu = \omega/\sqrt{q},\ \nu = p/\sqrt{q}). \qquad (15)$$

We have from (15) that K(0) = 1, $K(\infty) = 0$ for any ν. If $\nu \geq \sqrt{2}$, i.e., $p \geq \sqrt{2q}$, then $K(\mu)$ decreases for all μ. If $\nu < \sqrt{2}$, i.e. $p < \sqrt{2q}$, then $K(\mu)$ has its maximum at $\mu = 1 - \nu^2/2$.

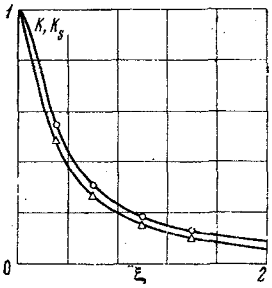

Fig. 1

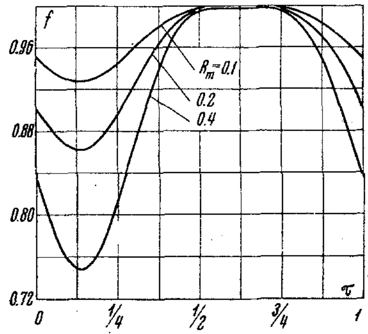

Fig. 2

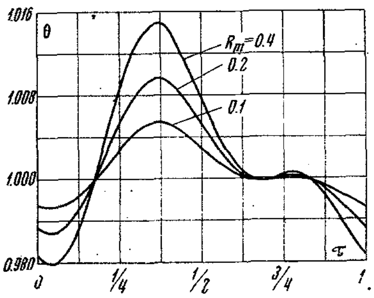

Fig. 3

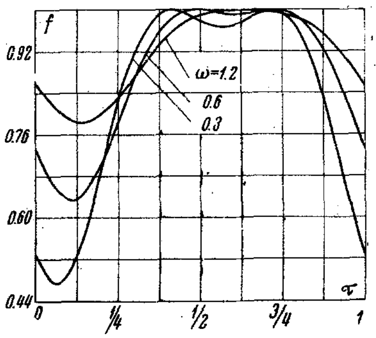

Fig. 4

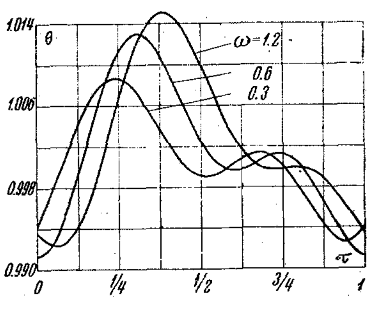

Fig. 5

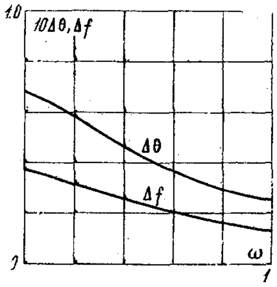

Fig. 6

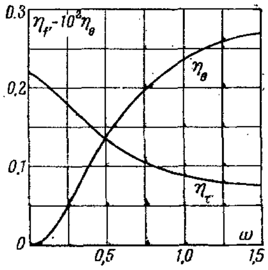

Fig. 7

Then for $p \geq \sqrt{2q}$ (high resistance coefficient) there is no resonance, while $p < \sqrt{2q}$ allows resonant oscillations of frequency $\omega_* = \sqrt{q - p^2/2}$, which corresponds to a gain

$$K_* = \frac{1}{\nu}\left(1 - {}^1\!/_4\nu^2\right)^{-1/2} = \frac{q}{p}\left(q - {}^1\!/_4 p^2\right)^{-1/2}$$

This shows that $K_* \to \infty$ and $\omega_* \to \sqrt{q}$ for $p \to 0$, i.e., the resonance will be more important for p small and for resonant frequencies close to $\sqrt{q}$.

It is readily seen from (11) and (12) that $p > \sqrt{2q}$ always for $R_m < 1$, so resonant oscillations do not occur in the system, and $K(\mu)$ decreases monotonically from 1 to 0 as μ goes from 0 to ∞.

These results apply also for finite $M_0 \neq 1$, since the p and q of (8) are not dependent on M_0, and so $p > \sqrt{2q}$ for $R_m < 1$, as for $M_0 \ll 1$. It can be shown that the oscillations will be nonresonant in this case also if the boundary conditions for the gasdynamic quantities are taken as $\theta = \mu = 1$ for $\xi = 0$ and $f = 1$ for $\xi = 1$. Allowance for the nonlinearity causes $K(\mu)$ to decrease more rapidly as μ increases, as Fig. 1 shows via $K(\mu)$ (circles) for $\nu^2 = 30$, $S = 0$ (linear approximation) and $K_s(\mu) = B_1/B_{10}$ (triangles) for $S = 0.08$ and $\nu^2 = 30$, in which B_1 is the amplitude for the first harmonic in the solution of (10), while B_{10} is the value at $\omega = 0$. Then $K_s(\mu) = K(\mu)$ when $S = 0$.

Numerical calculations have been performed; Figs. 2–7 show some results for $M_0 \ll 1$. Figures 2–5 illustrate the changes in θ and f during one period at the exit ($\xi = 1$) for $E_0 = 1$, $r = 0.5$, $P = 0.2$, $\varkappa = 5/3$, and various R_m and ω. The oscillations in the system parameters are not sinusoidal, although the driving force is.

Figure 6 shows $\Delta\theta$ and Δf (the scales of the variations from maximum to minimum for $\theta(\tau)$ and $f(\tau)$) as functions of ω for $E_0 = 1$, $r = 0.5$, $P = 0.2$, $R_m = 0.4$, $\varkappa = 5/3$. The two quantities decrease monotonically, because the oscillations are not resonant. Figure 7 illustrates

$$\eta_\theta = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[1 - \theta(1, \tau)\right]d\tau,$$

$$\eta_f = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[1 - f(1, \tau)\right]d\tau,$$

the period-mean differences between θ at the entrance and exit to the channel (and the same for f) as functions of ω for the same E_0, r, P, R_m and $\varkappa$. This means that η_0 is negative throughout this frequency range for the given E_0, r, P, R_m and $\varkappa$, i.e., the average gas temperature at the exit is higher than that at the inlet, and so the gas receives energy from the electromagnetic field source.

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